Auto-correlation (Wiener-Khinchin's theorem)

The autocorrelation, $R(x)$, of a function, $f(x)$, is defined as

$$R(x) \equiv \int_{-\infty}^{\infty} f(y) f(y+x) dy. \tag{1}$$

Recalling that the convolution between $f(x)$ and $g(x)$ is defined as

$$f(x) * g(x) \equiv \int_{-\infty}^{\infty} f(x-y) g(y) dy \tag{2}$$

it can be shown, by setting $g(x) \equiv f(-x)$, that

$$R(x) = f(x) * f(-x). \tag{3}$$

According to the Fourier convolution theorem,

$$\mathscr{F}(f(x) * g(x)) = F(\omega) G(\omega). \tag{4}$$

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(x) e^{-i\omega x} dx && (5) \\ &= \int_{-\infty}^{\infty} f(-x) e^{-i\omega x} dx && (6) \\ &= \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx && (7) \\ &= \overline{F(\omega)}. && (8) \end{aligned}$$

It then follows

$$\begin{aligned} \mathscr{F}(R(x)) &= F(\omega)\overline{F(\omega)} && (9) \\ &= |F(\omega)|^2 && (10) \\ &= P(\omega). && (11) \end{aligned}$$

i.e., the Fourier transform of the autocorrelation function is the power spectrum, $P(\omega)$. This is known as the *Wiener-Khinchin's theorem*.
Using the autocorrelation function to obtain the power spectrum is preferred over the direct Fourier transform as most of the signals have very narrow bandwidth.
If $P(\omega)$ is independent of the frequency, it is called white noise. If $P(\omega)$ is proportional to $1/f (= 1/\omega)$, the original data is called *pink noise*, $1/f$ *noise* or *fractal noise* and if $P(\omega)$ is proportional to $1/f^2$, it is called *brown noise*.

Application to Differential Equations

Diffusion equation

The governing equation for the diffusion equation (transient heat conduction) is

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \tag{12}$$

with the initial condition of

$$u(x,0) = f(x), \tag{13}$$

and x is defined over the entire range of $(-\infty, \infty)$. Because of the range given, Fourier transforms should be employed. By Fourier transforming eq.(12), one can obtain

$$\frac{\partial U(\omega,t)}{\partial t} = (i\omega)^2 U(\omega,t), \tag{14}$$

where $U(\omega)$ is the Fourier transform of $u(x,t)$. Equation (14) can be solved as

$$U(\omega,t) = A(\omega)\exp(-\omega^2 t). \tag{15}$$

The inverse Fourier transform of eq.(15) is

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega,t)e^{i\omega x}d\omega \tag{16}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\omega)\exp(-\omega^2 t)e^{i\omega x}d\omega. \tag{17}$$

The initial condition of eq.(13) at $t = 0$ is now incorporated as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\omega)e^{i\omega x}d\omega, \tag{18}$$

which implies that $A(\omega)$ is the Fourier transform of $f(x)$,i.e.

$$A(\omega) = F(\omega). \tag{19}$$

Hence the solution to eq.(12) with the initial condition of eq.(13) in the frequency domain is

$$U(\omega,t) = F(\omega)\exp(-\omega^2 t). \tag{20}$$

Using the Fourier convolution theorem, eq.(20) can be inverted as

$$u(x,t) = f(x) * F^{-1}\left(\exp(-\omega^2 t)\right). \tag{21}$$

Enter `Exp[-`w^2` t]` and `w` in the boxes above and press the button. Verify that you get

$$F^{-1}\left(\exp(-\omega^2 t)\right) = \frac{1}{2\sqrt{\pi t}}\exp\left(-\frac{x^2}{4t}\right). \tag{22}$$

so

$$u(x,t) = \left(\frac{1}{2\sqrt{\pi t}}\right)\exp(-\frac{x^2}{4t}) * f(x) \tag{23}$$
$$= \left(\frac{1}{2\sqrt{\pi t}}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{(x-y)^2}{4t}\right) f(y)dy. \tag{24}$$

If the initial condition is given by the Dirac delta function as

$$f(x) = \delta(x), \tag{25}$$

eq.(24) becomes

$$u(x,t) = \left(\frac{1}{2\sqrt{\pi t}}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{(x-y)^2}{4t}\right)\delta(y)dy \tag{26}$$
$$= \left(\frac{1}{2\sqrt{\pi t}}\right)\exp\left(-\frac{x^2}{4t}\right). \tag{27}$$

The animated graph below shows how diffusion process progresses as time goes by.

Fourier series solution

If the interval is finite, say $-\pi < x < \pi$, instead of $(-\infty, \infty)$, the solution can be expressed by the Fourier series instead of the Fourier transforms as

$$u(x,t) = \sum_{m=-\infty}^{\infty} u_m e^{imx}, \tag{28}$$

where u_m are unknown Fourier coefficients.
Noting that

$$u''(x,t) = \sum_{m=-\infty}^{\infty} (im)^2 u_m e^{imx}, \tag{29}$$

and

$$\frac{\partial u}{\partial t} = \sum_{m=-\infty}^{\infty} \frac{du_m}{dt} e^{imx}, \tag{30}$$

so the governing equation of $\partial u/\partial t = \partial u^2/\partial x^2$ becomes

$$\sum_{m=-\infty}^{\infty} \frac{du_m}{dt} e^{imx} = \sum_{m=-\infty}^{\infty} (-m^2) u_m e^{imx}, \tag{31}$$

or

$$\frac{du_m}{dt} = -m^2 u_m, \tag{32}$$

which can be solved as

$$u_m = A_m \exp(-m^2 t), \tag{33}$$

thus

$$\begin{aligned} u(x,t) &= \sum_{m=-\infty}^{\infty} u_m e^{imx} & (34) \\ &= \sum_{m=-\infty}^{\infty} A_m \exp(-m^2 t) e^{imx}. & (35) \end{aligned}$$

The initial condition that at $t = 0$, $u = f(x)$ is now incorporated as

$$f(x) = \sum_{m=-\infty}^{\infty} A_m e^{imx}, \tag{36}$$

which implies that A_m is the Fourier coefficient of $f(x)$ so

$$\begin{aligned} A_m &= f_m & (37) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} dx, & (38) \end{aligned}$$

thus

$$u(x,t) = \sum_{m=-\infty}^{\infty} f_m \exp(-m^2 t) e^{imx}. \tag{39}$$

Convolution theorem for Fourier series

(Definition: Fourier series convolution)

$$f * g \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)g(y)dy \tag{40}$$

(Convolution theorem)

$$f_m g_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f * g e^{-imx}dx, \tag{41}$$

or

$$f * g = \sum_{m=-\infty}^{\infty} f_m g_m e^{imx} \tag{42}$$

x	C_m
$f(x)$	f_m
$g(x)$	g_m
$f * g$	$f_m g_m$

Parseval's theorem for Fourier series

The Fourier and inverse Fourier transform formulas are given as

$$f(x) = \sum_{m=-\infty}^{\infty} C_m e^{imx}, \tag{43}$$

$$C_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}dx, \tag{44}$$

from which

$$f(x)\overline{f(x)} = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} C_m\overline{C_n}e^{imx}e^{-inx} \tag{45}$$

Integrating the both sides gives

$$\int_{-\pi}^{\pi} f(x)\overline{f(x)}dx = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} C_m\overline{C_n}\int_{-\pi}^{\pi} e^{imx}e^{-inx}dx \tag{46}$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} C_m\overline{C_n}\int_{-\pi}^{\pi} e^{i(m-n)x}dx \tag{47}$$

$$= 2\pi\sum_{m=-\infty}^{\infty} |C_m|^2, \tag{48}$$

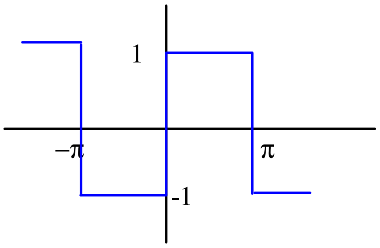
where

$$\int_{-\pi}^{\pi} e^{imx}e^{-inx}dx = \begin{cases} 0 & m \neq n \\ 2\pi & m = n \end{cases}$$

was used.
Hence the Parseval's theorem for Fourier series is stated as

$$\int_{-\pi}^{\pi}\{f(x)\}^2 dx = 2\pi\sum_{m=-\infty}^{\infty}|C_m|^2. \tag{49}$$

Example



$$C_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}dx \tag{50}$$

$$= \frac{1}{2\pi}\left(\int_{-\pi}^{0}(-1)e^{-imx}dx + \int_{0}^{\pi}(+1)e^{-imx}dx\right) \tag{51}$$

$$= \frac{1}{\pi}\left(\frac{(-1)^m - 1}{m}\right)i, \tag{52}$$

so

$$|C_m|^2 = \frac{((-1)^m - 1)^2}{\pi^2 m^2}, \tag{53}$$

or

$$|C_m|^2 = \begin{cases} \frac{4}{\pi^2}\frac{1}{(2m-1)^2} & m = \pm 1, \pm 3, \pm 5, \ldots \\ 0 & m = 0, \pm 2, \pm 4, \pm 6, \ldots \end{cases} \tag{54}$$

Hence, eq.(49) becomes

$$2\pi = 2\pi \sum_{m=-\infty}^{\infty} \frac{4}{\pi^2}\frac{1}{(2m-1)^2} \tag{55}$$

from which one obtains

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}. \tag{56}$$